Journal of Sound and Vibration (2001) **240**(2), 203–217
doi:10.1006/jsvi.2000.3213, available online at http://www.idealibrary.com on IDEAL®

ON THE RECONSTRUCTION OF A DAMPED VIBRATING SYSTEM FROM TWO COMPLEX SPECTRA, PART 1: THEORY

G. M. L. GLADWELL

Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
E-mail: ggladwell@uwaterloo.ca

(*Received* 22 *November* 1999, *and in final form* 3 *April* 2000)

The paper concerns an n-degree of freedom damped vibrating system consisting of $n-1$ masses connected in parallel, by springs and dampers, to an nth mass. The paper analyzes the construction of such a system from the given complex eigenvalue data. The analysis has two parts: the establishment of the conditions on the eigenvalues which ensure that they correspond to an actual system; the derivation of the system parameters from the eigenvalues.

1. INTRODUCTION

This paper concerns a linear vibrating system, with n degrees of freedom, governed by an equation of the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t), \tag{1}$$

where $\cdot = \mathrm{d}/\mathrm{d}t$. The substitutions

$$\mathbf{u}(t) = \mathbf{u}\mathrm{e}^{\lambda t}, \quad \mathbf{f}(t) = \mathbf{f}\mathrm{e}^{\lambda t} \tag{2}$$

lead to the equation

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{u} = \mathbf{f}. \tag{3}$$

We will consider the case in which $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are symmetric, $\mathbf{M}$ and $\mathbf{K}$ are positive-definite (p–d) and $\mathbf{C}$, positive semi-definite (ps–d).

The free vibration of the system is governed by the equation

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{u} = \mathbf{0}. \tag{4}$$

The values of λ for which this equation has a non-trivial solution, form the *complex spectrum* of the *quadratic pencil*

$$\mathbf{Q}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}. \tag{5}$$

The values of λ in the spectrum appear in pairs, n pairs in all: real negative pairs, corresponding to *overdamped* modes; complex conjugate pairs corresponding to

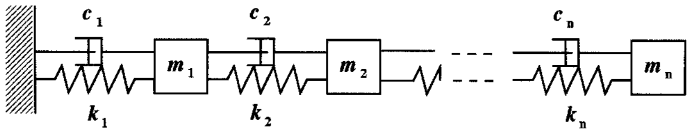

Figure 1. A series system of in-line damped vibrators.

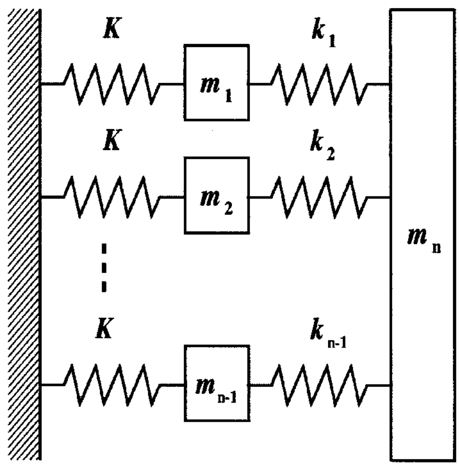

Figure 2. An undamped parallel system of vibrators.

underdamped modes; or complex conjugate imaginary pairs corresponding to *undamped* modes.

We consider a *second spectrum* of equation (5), consisting of these values of λ for which equation (4) has a non-trivial solution having $u_n = 0$. This spectrum will consist of $(n-1)$ pairs, again of the three possible kinds. This spectrum is that for the *truncated* pencil,

$$Q_L(\lambda) = \mathbf{M}_L\lambda^2 + \mathbf{C}_L\lambda + \mathbf{K}_L, \tag{6}$$

where $\mathbf{M}_L$, $\mathbf{C}_L$, $\mathbf{K}_L$ are obtained from $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, respectively, by deleting the nth row and column of each of the three matrices.

The $2n-1$ pairs of eigenvalues contained in the spectra of $\mathbf{Q}(\lambda)$ and $\mathbf{Q}_L(\lambda)$ are clearly insufficient to determine the $3n(n+1)/2$ coefficients in the three matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$. To obtain a unique solution, or a manageable family of solutions, we must drastically constrain the form of the matrices. Figures 1 and 2 show two possible systems. The one shown in Figure 1 is an in-line set of masses $(m_i)_1^n$ connected by springs $(k_i)_1^n$ and dampers $(c_i)_1^n$; shown in Figure 2 is a parallel system of masses $(m_i)_1^{n-1}$ all connected both to ground and to the mass m_n, by springs and dampers.

The outline of the paper is as follows. In section 2, we summarize the known results for the undamped version of the system in Figure 1; the results for the damped version are given in section 3. Then in the sections 4 and 5 we analyze the undamped and damped versions of the parallel system shown in Figure 2.

2. THE UNDAMPED SERIES SYSTEM

For an undamped system, each spectrum consists of complex conjugate imaginary pairs, $\pm\,\mathrm{i}\omega_j$, $j = 1, 2, \ldots, n$, for the unconstrained system; $\pm\,\mathrm{i}\sigma_j$, $j = 1, 2, \ldots, n-1$ for the constrained system. For this case, it is convenient to write

$$\lambda_j = \omega_j^2, \quad \mu_j = \sigma_j^2. \tag{7}$$

Since the μ_j are the squares of the natural frequencies of the constrained system, they must satisfy the interfacing conditions

$$0 \leqslant \lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n. \tag{8}$$

If the system is grounded, i.e., $k_1 > 0$, and connected, i.e. $(k_i)_2^n > 0$, then all the inequalities in equation (8) are strict, i.e.,

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n. \tag{9}$$

We will consider only this case.

The time-reduced equation governing the free vibrations of the undamped system is

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0}, \tag{10}$$

where

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & \\ -k_2 & k_2 + k_3 & -k_3 \\ \cdots & \cdots & \cdots \\ & -k_n & k_n \end{bmatrix}, \qquad \mathbf{M} = diag(m_1, m_2, \ldots, m_n). \tag{11}$$

The substitutions

$$\mathbf{M} = \mathbf{M}^{1/2} \cdot \mathbf{M}^{1/2}, \quad \mathbf{M}^{1/2}\mathbf{u} = \mathbf{x}, \quad \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2} = \mathbf{A} \tag{12}$$

reduce equation (10) to the standard form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{13}$$

The matrix **A** is symmetric, tridiagonal, with negative co-diagonal.

Gantmakher and Krein [1] first solved the basic problem of reconstructing **A** from $(2n-1)$ quantities $(\lambda_j)_1^n$ and $(\mu_j)_1^{n-1}$. Golub and Boley [2] gave a stable numerical algorithm for constructing **A**. In the vast literature related to the problem and its generalizations (see Gladwell [3–5]), we mention Gladwell and Willms [6], Ram and Gladwell [7] and Ram [8]. Gladwell [9] showed how to construct an isospectral family of (undamped) systems like that in equation (10) which had just one given spectrum $(\lambda_j)_1^n$. Gladwell [10] generalized the problem to finite element method (FEM) models with tridiagonal, rather than simply diagonal, mass matrix.

3. THE DAMPED SERIES SYSTEM

The quadratic pencil corresponding to the damped series system of Figure 1 is equation (5), where **K**, **M** are given by equation (11) and

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & & \\ -c_2 & c_2 + c_3 & -c_3 & \\ \cdots & \cdots & \cdots & \\ & & -c_n & c_n \end{bmatrix}. \tag{14}$$

The substitutions

$$\mathbf{M}=\mathbf{M}^{1/2}\cdot\mathbf{M}^{1/2},\quad \mathbf{M}^{1/2}\mathbf{u}=\mathbf{x},\quad \mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2}=\mathbf{B},\quad \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}=\mathbf{A} \tag{15}$$

reduce equation (4) to

$$(I\lambda^2+\mathbf{B}\lambda+\mathbf{A})\mathbf{x}=\mathbf{0}. \tag{16}$$

Now both **A**, **B** are symmetric, tridiagonal, p–d and ps–d matrices respectively. Ram and Elhay [11] studied the reconstruction of **A** and **B** from two spectra. The basic step in their reconstruction is a generalization of the basic step used in some of the early papers, e.g., Hald [12], in the undamped case, as we now describe.

In the undamped case there is just one matrix **A**, and it has the form

$$\mathbf{A}=\begin{bmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & -b_2 & \\ \cdots & \cdots & \cdots & \\ & & -b_{n-1} & a_n \end{bmatrix}. \tag{17}$$

The principal minors $P_r(\lambda)$ of $\mathbf{A}-\lambda\mathbf{I}$ form a Sturm sequence with initial values

$$P_0(\lambda)=1,\quad P_1(\lambda)=a_1-\lambda, \tag{18}$$

and recurrence relation

$$P_r(\lambda)=(a_r-\lambda)P_{r-1}(\lambda)-b_{r-1}^2\,P_{r-2}(\lambda). \tag{19}$$

The given data, $(\lambda_j)_1^n$ and $(\mu_j)_1^{n-1}$ are the zeros of $P_n(\lambda)$ and $P_{n-1}(\lambda)$ respectively. Thus,

$$P_n(\lambda)=\prod_{j=1}^{n}(\lambda_j-\lambda),\quad P_{n-1}(\lambda)=\prod_{j=1}^{n-1}(\mu_j-\lambda). \tag{20}$$

Now by considering equation (19) with $r=n$, and knowing $P_n(\lambda)$ and $P_{n-1}(\lambda)$, we can find $P_{n-2}(\lambda)$, a_n and b_{n-1} by synthetic division. Having found $P_{n-2}(\lambda)$, we repeat this step to find successively $a_{n-1}, b_{n-2};\ \ldots;\ a_2, b_1;\ a_1$.

The generalization to the damped case is as follows. The matrix **B** of equation (16) has the form

$$\mathbf{B}=\begin{bmatrix} d_1 & -e_1 & & \\ -e_1 & d_2 & -e_2 & \\ \cdots & \cdots & \cdots & \cdots \\ & & -e_{n-1} & d_n \end{bmatrix}. \tag{21}$$

The polynomials corresponding to $P_1(\lambda)$ of equation (18) are

$$P_0(\lambda)=1,\quad P_1(\lambda)=\lambda^2+d_1\lambda+a_1, \tag{22}$$

and the recurrence relation is now

$$P_r(\lambda)=(\lambda^2+d_r\lambda+a_r)P_{r-1}(\lambda)-(e_{r-1}\lambda+b_{r-1})^2P_{r-2}(\lambda). \tag{23}$$

Again the given data $(\lambda_j)_1^{2n}$ and $(\mu_j)_1^{2n-2}$ (which appear as complex conjugate or real pairs, as we noted earlier) are the zeros of $P_n(\lambda)$ and $P_{n-1}(\lambda)$, i.e.,

$$P_n(\lambda) = \prod_{j=1}^{2n} (\lambda_j - \lambda), \quad P_{n-1}(\lambda) = \prod_{j=1}^{2n-2} (\mu_j - \lambda). \tag{24}$$

The essential contribution which Ram and Elhay made was an algorithm to carry out the synthetic divisions needed to compute a_r, d_r, e_{r-1}, b_{r-1} and $P_{r-2}(\lambda)$ from $P_r(\lambda)$ and $P_{r-1}(\lambda)$. The main difficulty which they encounter is that of not knowing the conditions which the two complex spectra must satisfy to ensure that the pencil is real and the matrices p–d and ps–d. We will encounter this difficulty in our analysis given below, but it will occur in a somewhat milder form.

4. THE UNDAMPED PARALLEL SYSTEM

We suppose that two spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ are given, and that they satisfy the strict interlacing condition (9). We construct a system, shown in Figure 2, which has these two spectra. Equation (10) has the form

$$\begin{bmatrix} K + k_1 - m_1\lambda & & & -k_1 \\ & K + k_2 - m_2\lambda & & -k_2 \\ \cdots & \cdots & \cdots & \cdots \\ & & K + k_{n-1} - m_{n-1}\lambda & -k_{n-1} \\ -k_1 & -k_2 & -k_{n-1} & k - m_n\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_{n-1} \\ u_n \end{bmatrix} = \mathbf{0}, \tag{25}$$

where

$$k = \sum_{j=1}^{n-1} k_j. \tag{26}$$

The eigenvalues of the constrained system are

$$\mu_j = (K + k_j)/m_j, \quad j = 1, 2, \ldots, n-1. \tag{27}$$

When reduced to the standard form, equation (25) is

$$(A - \lambda I)x \equiv \begin{bmatrix} \mu_1 - \lambda & & & -b_1 \\ & \mu_2 - \lambda & & -b_2 \\ \cdots & \cdots & \cdots & \cdots \\ & & \mu_{n-1} - \lambda & -b_{n-1} \\ -b_1 & -b_2 & -b_{n-1} & a_n - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{0}, \tag{28}$$

where

$$b_j = k_j/(m_j m_n)^{1/2}, \quad a_n = k/m_n. \tag{29}$$

The trace of the reduced matrix $\mathbf{A}$ is

$$\sum_{j=1}^{n-1} \mu_j + a_n = \sum_{j=1}^{n} \lambda_j, \tag{30}$$

which yields a_n. Now the first $(n-1)$ lines of equation (28) give

$$(\mu_j - \lambda) x_j = b_j x_n \tag{31}$$

which, when substituted into the last line, gives the eigenvalue equation

$$f(\lambda) \equiv -\sum_{j=1}^{n-1} \frac{b_j^2}{\mu_j - \lambda} + a_n - \lambda = 0. \tag{32}$$

The zeros and poles of $f(\lambda)$ are $(\lambda_j)_1^n$ and $(\mu_j)_1^{n-1}$, so that

$$f(\lambda) = \prod_{j=1}^{n} (\lambda_j - \lambda) \Big/ \prod_{j=1}^{n-1} (\mu_j - \lambda) \tag{33}$$

and hence

$$-b_j^2 = \prod_{i=1}^{n} (\lambda_i - \mu_j) \Big/ {\prod_{i=1}^{n-1}}' (\mu_i - \mu_j), \tag{34}$$

where $'$ denotes $i \neq j$. The strict interlacing condition (9) implies $b_j^2 > 0$.

We have now identified the reduced matrix $\mathbf{A}$: the first $(n-1)$ diagonal elements are data, the last is given by equation (30); and the bordering elements are given by equation (34).

Now we must construct the masses and stiffnesses; we do that by reversing equations (26), (27) and (29). Put $m_1 = 1$, $m_j = y_j^2$, $j = 1, 2, \ldots, n-1$, then equation (29) gives $k_j = b_j y_j$. Then equation (27), and equations (26) and (29), respectively, give

$$\mu_j y_j^2 - b_j y_j - K = 0, \quad j = 1, 2, \ldots, n-1, \tag{35}$$

$$-\sum_{j=1}^{n-1} b_j y_j + a_n = 0. \tag{36}$$

Equation (35) is a quadratic equation for y_j with just one positive root:

$$y_j = \frac{b_j + \sqrt{b_j^2 + 4K\mu_j}}{2\mu_j}. \tag{37}$$

When substituted into equation (36), this yields an equation for K:

$$g(K) \equiv -\sum_{j=1}^{n} b_j \frac{\{b_j + \sqrt{b_j^2 + 4K\mu_j}\}}{2\mu_j} + a_n = 0. \tag{38}$$

The function $g(K)$ is monotonically decreasing, $g(K) \to -\infty$ as $K \to \infty$, and $g(0) = f(0)$, where $f(\lambda)$ is given by equation (32). Now, $f(\mu_1 -) < 0$ and $f(\lambda_1) = 0$, $0 < \lambda_1 < \mu_1$, imply $f(0) > 0$. Thus, $g(K)$ has just one positive root K. Having found K we may find y_j from equation (37) and complete the reconstruction of the system: $k_j = b_j y_j$, $m_j = y_j^2$.

We have now constructed a unique system of the form shown in Figure 2, which has the desired spectra. This unique system is a particular member of the family shown in Figure 3. For the given reduced matrix shown in equation (28), the equations corresponding to equations (35) and (36) are now

$$\mu_j y_j^2 - b_j y_j - K_j = 0, \quad j = 1, 2, \ldots, n-1, \tag{39}$$

$$-\sum_{j=1}^{n-1} b_j y_j + a_n = k_n. \tag{40}$$

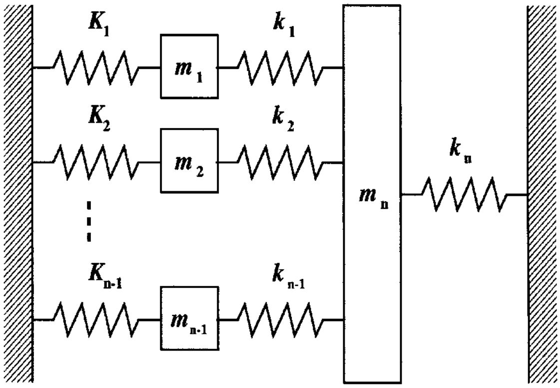

Figure 3. A more general undamped parallel system of n vibrators.

Here $(\mu_j, b_j)_1^{n-1}$ and a_n are known and $(y_j, K_j)_1^{n-1}$ and k_n are unknown. Equation (39) gives

$$y_j = \frac{b_j + \sqrt{b_j^2 + 4\mu_j K_j}}{2\mu_j}, \tag{41}$$

and thus equation (40) becomes

$$-\sum_{j=1}^{n-1} b_j \frac{(b_j + \sqrt{b_j^2 + 4\mu_j K_j})}{2\mu_j} + a_n = k_n. \tag{42}$$

The left-hand side is a function $g(K_1, K_2, \ldots, K_{n-1})$ and, as before $g(0, 0, \ldots, 0) > 0$. Since g is continuous at $(0, 0, \ldots, 0)$, there is a neighbourhood of $(0, 0, \ldots, 0)$ in which $g(K_1, \ldots, K_{n-1}) > 0$. Thus there always exists an infinite family of solutions to equation (42) corresponding to positive $(K_j)_1^{n-1}$ and positive k_n. Once we have chosen one such solution we may form $(y_j)_1^{n-1}$ from equation (41) and then $(k_j, m_j)_1^{n-1}$ as before.

5. THE DAMPED PARALLEL SYSTEM

We consider the system shown in Figure 4. Equation (4) is

$$\begin{bmatrix} F_1(\lambda) & & & & -(g_1\lambda + k_1) \\ & F_2(\lambda) & & & -(g_2\lambda + k_2) \\ & & \ddots & & \vdots \\ & & & F_{n-1}(\lambda) & -(g_{n-1}\lambda + k_{n-1}) \\ -(g_1\lambda + k_1) & -(g_2\lambda + k_2) & \ldots & -(g_{n-1}\lambda + k_{n-1}) & F_n(\lambda) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{43}$$

where

$$\begin{aligned} F_j(\lambda) &= m_j\lambda^2 + (g_j + G_j)\lambda + k_j + K_j, \quad j = 1, 2, \ldots, n-1, \\ F_n(\lambda) &= m_n\lambda^2 + g\lambda + k, \quad g = \sum_{j=1}^{n} g_j, \quad k = \sum_{j=1}^{n} k_j. \end{aligned} \tag{44}$$

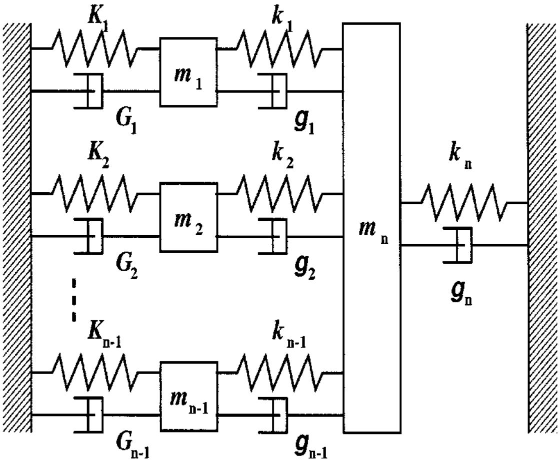

Figure 4. A damped parallel system of n vibrators.

We suppose that the system is connected, so that $(k_j)_1^n > 0$, $(K_j)_1^n > 0$. We now reduce equation (43) to the standard form

$$\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0}, \tag{45}$$

where

$$\mathbf{Q}(\lambda) = \begin{bmatrix} \mathbf{Q}_L(\lambda) & -\mathbf{a}\lambda - \mathbf{b} \\ -\mathbf{a}\lambda - \mathbf{b} & \lambda^2 + c_n\lambda + \sigma_n^2 \end{bmatrix} = \lambda^2\mathbf{I}_n + \lambda\mathbf{A} + \mathbf{B}, \tag{46}$$

$$\mathbf{Q}_L(\lambda) = \lambda^2\mathbf{I}_{n-1} + \lambda\mathbf{C} + \sum{}^2, \tag{47}$$

$$\mathbf{C} = diag(c_1, c_2, \ldots, c_{n-1}), \quad \sum{}^2 = diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2), \tag{48}$$

$$c_j = \frac{g_j + G_j}{m_j}, \quad \sigma_j^2 = \frac{k_j + K_j}{m_j}, \quad a_j = \frac{g_j}{(m_j m_n)^{1/2}}, \quad b_j = \frac{k_j}{(m_j m_n)^{1/2}},$$

$$j = 1, 2, \ldots, n-1. \tag{49}$$

$$c_n = \frac{g}{m_n}, \quad \sigma_n^2 = \frac{k}{m_n}. \tag{50}$$

The $2n$ eigenvalues of the real symmetric quadratic pencil $\mathbf{Q}(\lambda)$ occur in pairs, either real or complex conjugate. The physically important case is when they all lie in the left-hand half of the complex plane. We assume therefore that there are $r (0 \leqslant r \leqslant n)$ real pairs λ_j, λ'_j, where $\lambda_j < 0, \lambda'_j < 0, j = 1, 2, \ldots, r$, and $n - r$ complex conjugate pairs $\lambda_j, \overline{\lambda}_j$, where $\mathrm{Re}(\lambda_j) \leqslant 0$, $\mathrm{Im}(\lambda_j) > 0, j = r+1, \ldots, n$. The first r pairs correspond to the overdamped modes, and the remainder to the undamped ($\mathrm{Re}(\lambda_j) = 0$) or underdamped ($\mathrm{Re}(\lambda_j) < 0$) modes.

We label the eigenvalues of $\mathbf{Q}_L(\lambda)$ similarly: μ_j, μ'_j, where $\mu_j < 0, \mu'_j < 0, j = 1, 2, \ldots, s$; $\mu_j, \bar{\mu}_j$, where $\mathrm{Re}(\mu_j) \leqslant 0$, $\mathrm{Im}(\mu_j) > 0, j = s+1, \ldots, n-1$.

We now choose $(c_j, \sigma_j^2)_1^{n-1}$ so that

$$\lambda^2 + c_j\lambda + \sigma_j^2 = \begin{cases}(\lambda - \mu_j)(\lambda - \mu_j'), & j = 1, 2, \ldots, s,\\ (\lambda - \mu_j)(\lambda - \bar{\mu}_j), & j = s + 1, \ldots, n - 1.\end{cases} \tag{51}$$

This means that $\mathbf{Q}_L(\lambda)$ has the specified eigenvalues.

Now we must choose the remaining quantities $(a_j, b_j)_1^{n-1}$, c_n and σ_n^2 so that

$$det(\mathbf{Q}(\lambda)) = \prod_{j=1}^{n} (\lambda^2 + d_j\lambda + \omega_j^2), \tag{52}$$

where

$$\lambda^2 + d_j\lambda + \omega_j^2 = \begin{cases}(\lambda - \lambda_j)(\lambda - \lambda_j'), & j = 1, 2, \ldots, r,\\ (\lambda - \lambda_j)(\lambda - \bar{\lambda}_j), & j = r + 1, \ldots, n.\end{cases} \tag{53}$$

The determinant of $\mathbf{Q}(\lambda)$, as given by equation (46), is

$$det(\mathbf{Q}(\lambda)) = \prod_{j=1}^{n} (\lambda^2 + c_j\lambda + \sigma_j^2) - \sum_{j=1}^{n-1} (a_j\lambda + b_j)^2 \prod_{k=1}^{n-1}{}' (\lambda^2 + c_k\lambda + \sigma_k^2), \tag{54}$$

where $'$ denotes $k \neq j$. By equating the constant terms in equations (52) and (54) we deduce

$$\left(\prod_{j=1}^{n-1} \sigma_j^2\right)\sigma_n^2 - \sum_{j=1}^{n-1} b_j^2 \prod_{k=1}^{n-1}{}' \sigma_k^2 = \prod_{j=1}^{n} \omega_j^2, \tag{55}$$

while by equating the coefficients of λ^{2n-1} we obtain

$$\sum_{j=1}^{n-1} c_j + c_n = \sum_{j=1}^{n} d_j. \tag{56}$$

We comment on these equations. The quantities $(\sigma_k^2)_1^{n-1}$ and $(\omega_j^2)_1^n$ may be computed, via equations (51) and (53), respectively, from the data. Once real $(b_j)_1^{n-1}$ are known, equation (55) gives σ_n^2, and ensures that $\sigma_n^2 > 0$. Equation (56) gives c_n in terms of $(d_j)_1^n$ and $(c_j)_1^{n-1}$, which again are given, via equations (53) and (51), respectively, in terms of the data.

For convenience, we will call the complete system S, and the system constrained so that $u_n = 0, S_L$. We note that if S is undamped then S_L is undamped. For the stated contraints on the eigenvalues $\lambda_j, \lambda_j', \mu_j, \mu_j'$ mean that all c_j, d_j are non-negative. If S is undamped, i.e., $(d_j)_1^n = 0$, then equation (56) implies $(c_j)_1^n = 0$. If $g_n = 0$, then the converse is true: if S_L is undamped then S is undamped. For if S_L is undamped, then $(c_j)_1^{n-1} = 0$, so that, from equation (49), $g_j + G_j = 0, j = 1, 2, \ldots, n - 1$. Therefore, since $g_j \geqslant 0$, $G_j \geqslant 0$, we must have $(g_j, G_j)_1^{n-1} = 0$, and thus $g = 0$ and $c_n = 0$. Now $d_j \geqslant 0$ and equation (56) implies $(d_j))_1^n = 0$; S is undamped.

Now we must find the $(2n - 2)$ quantities $(a_j, b_j)_1^{n-1}$. We obtain them by equating equations (52) and (54) for the s pairs $(\mu_j, \mu_j')_1^s$ and for the $n - 1 - s$ pairs $(\mu_j, \bar{\mu}_j)_{s+1}^{n-1}$. First we proceed formally. We have

$$(a_j\mu_j + b_j)^2 = \{R(\mu_j)\}^2, \quad (a_j\mu_j' + b_j)^2 = \{R(\mu_j')\}^2, \quad j = 1, \ldots, s, \tag{57}$$

$$(a_j\mu_j + b_j)^2 = \{R(\mu_j)\}^2, \quad (a_j\bar{\mu}_j + b_j)^2 = \{R(\bar{\mu}_j)\}^2, \quad j = s + 1, \ldots, n - 1, \tag{58}$$

where

$$\{R(\mu)\}^2 = \frac{-\prod_{k=1}^{n}(\mu^2 + d_k\mu + \omega_k^2)}{\prod_{k=1}^{(n-1)\prime}(\mu^2 + c_k\mu + \sigma_k^2)} = \frac{N(\mu)}{D(\mu)}. \tag{59}$$

Note that $R(\mu)$ is evaluated for one of $\mu_j, \mu'_j, \bar{\mu}_j$; $'$ denotes $k \neq j$.

The two pairs (57) and (58) are fundamentally different. First, consider equation (57); μ_j, μ'_j are real and negative so that, for a real solution a_j, b_j with $a_j \geqslant 0, b_j > 0$, $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ must be finite and non-negative. In this case, i.e., $1 \leqslant j \leqslant s$, the numerator and denominator of $\{R(\mu_j)\}^2$ are

$$N(\mu_j) = -\prod_{k=1}^{r}(\mu_j - \lambda_k)(\mu_j - \lambda'_k)\cdot\prod_{k=r+1}^{n}(\mu_j - \lambda_k)(\mu_j - \bar{\lambda}_k) = A(\mu_j)\cdot B(\mu_j), \tag{60}$$

$$D(\mu_j) = -\prod_{k=1}^{s\prime}(\mu_j - \mu_k)(\mu_j - \mu'_k)\cdot\prod_{k=s+1}^{n}(\mu_j - \mu_k)(\mu_j - \bar{\mu}_k) = C(\mu_j)\cdot E(\mu_j). \tag{61}$$

Similarly,

$$\{R(\mu'_j)\}^2 = N(\mu'_j)/D(\mu'_j) = A(\mu'_j)B(\mu'_j)/\{C(\mu'_j)E(\mu'_j)\}. \tag{62}$$

If $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ are to be finite, then $D(\mu_j), D(\mu'_j)$ must be non-zero, and thus the μ_j, μ'_j must be distinct. We order them so that

$$\mu_1 < \mu'_1 < \mu_2 < \mu'_2 < \cdots < \mu'_s < 0 \tag{63}$$

and we suppose that the $(\lambda_j, \lambda'_j)_1^r$ may also be so ordered, i.e.,

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \cdots < \lambda'_r < 0. \tag{64}$$

The quantities $B(\mu_j)$, $B(\mu'_j)$, $E(\mu_j)$, $E(\mu'_j)$, being squares of the norms of non-zero complex quantities, are positive. Thus, the conditions for $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ to be non-negative are that

$$A(\mu_j)C(\mu_j) \geqslant 0, \quad A(\mu'_j)C(\mu'_j) \geqslant 0. \tag{65}$$

We note that equation (63) implies $C(\mu_j) \neq 0$, $C(\mu'_j) \neq 0$. When written in full, equation (65) gives

$$-\prod_{k=1}^{r}(\mu_j - \lambda_k)(\mu_j - \lambda'_k)\cdot{\prod_{k=1}^{s}}'(\mu_j - \mu_k)(\mu_j - \mu'_k) \geqslant 0, \quad j = 1, 2, \ldots, s, \tag{66}$$

$$-\prod_{k=1}^{r}(\mu'_j - \lambda_k)(\mu'_j - \lambda'_k)\cdot{\prod_{k=1}^{s}}'(\mu'_j - \mu_k)(\mu'_j - \mu'_k) \geqslant 0, \quad j = 1, 2, \ldots, s. \tag{67}$$

It may be verified that the necessary and sufficient conditions for these inequalities to hold are that each μ_j lie between a λ_k, λ'_k pair, and each μ'_j also lie between a λ_l, λ'_l pair; these pairs may be the same or different, both for one pair μ_j, μ'_j and for two or more pairs μ_j, μ'_j, as shown in Figure 5. In particular, we note that if S_L has an overdamped pair μ_j, μ'_j, then S must have *at least one* overdamped pair also. When inequalities (66) and (67) are satisfied, then $\{R(\mu_j)\}^2$, $\{R(\mu'_j)\}^2$ are non-negative, and equations (57) may be replaced by

$$a_j\mu_j + b_j = \pm R(\mu_j), \quad a_j\mu'_j + b_j = \pm R(\mu'_j), \tag{68}$$

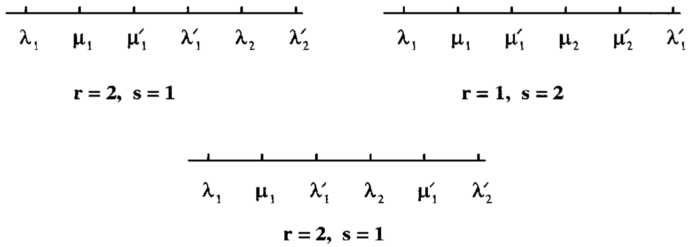

Figure 5. Three examples of possible μ, λ configurations for overdamped S_L eigenvalues.

where $R(\mu_j) = R$ and $R(\mu_j') = R'$ denote the non-negative square roots. In general, there will be four solutions for a_j, b_j; we may write these as

$$a_j^{(1)}, b_j^{(1)};\ a_j^{(2)}, b_j^{(2)};\ -a_j^{(1)},\ -b_j^{(1)};\ -a_j^{(2)},\ -b_j^{(2)}.$$

Suppressing index j, we have

$$a^{(1)} = \frac{R' + R}{\mu' + \mu'}, \quad b^{(1)} = \frac{-\mu' R - \mu R'}{\mu' - \mu}, \tag{69}$$

$$a^{(2)} = \frac{R' + R}{\mu' + \mu'}, \quad b^{(2)} = \frac{\mu' R - \mu R'}{\mu' - \mu}. \tag{70}$$

Since R, R' are both non-negative and $\mu < \mu' < 0$, the first solution has $a^{(1)} \geqslant 0, b^{(1)} \geqslant 0$; the inequalities will be strict unless $R = 0 = R'$. We note that one of R, R' can be zero, and will be zero when a μ_j or μ_j' $(j = 1, \ldots, s)$ equals a λ_k or λ_k' $(k = 1, 2, \ldots, r)$. We conclude that the necessary and sufficient condition for there to be a solution for $a_j^{(1)}, b_j^{(1)}$ with $a_j^{(1)} \geqslant 0$, $b_j^{(1)} > 0$ is that μ_j, μ_j' lie between a λ_k, λ_k' pair, and one of μ_j, μ_j' lies strictly between a λ_k, λ_k' pair. We note that if $a_j^{(1)} \geqslant 0$, $b_j^{(1)} > 0$, then $a_j^{(1)} > 0$. We note that the second solution may or may not be positive.

Now consider equation (58). Since μ_j is complex we may take the square roots of the complex quantities $\{R(\mu_j)\}^2$ and $\{R(\bar{\mu}_j)\}^2$ and get

$$a_j \mu_j + b_j = \pm R(\mu_j), \quad a_j \bar{\mu}_j + b_j = \pm \overline{R(\mu_j)}. \tag{71}$$

Now, for consistency we must take the same signs in the two equations. If we write

$$\{R(\mu_j)\}^2 = |R_j|^2 \mathrm{e}^{2\mathrm{i}\theta_j}, \quad 0 \leqslant \theta_j < \pi, \tag{72}$$

and take

$$\mu_j = \rho_j \mathrm{e}^{\mathrm{i}\alpha_j}, \quad \frac{\pi}{2} < \alpha_j < \pi, \tag{73}$$

then with $R(\mu_j) = |R_j| \mathrm{e}^{\mathrm{i}\theta_j}$ and the positive signs in equation (71) we find

$$a_j = \frac{R_j \sin\theta_j}{\rho_j \sin\alpha_j}, \quad b_j = \frac{R_j \sin(\alpha_j - \theta_j)}{\sin\alpha_j}. \tag{74}$$

If the system is to be a connected system then we must have $a_j \geqslant 0$ and $b_j > 0$. This means that we must have

$$0 \leqslant \theta_j < \alpha_j, \quad j = s + 1, \ldots, n - 1. \tag{75}$$

We have now found the conditions for the data to correspond to a real quadratic pencil. We must now investigate when and whether we can find real positive masses, stiffnesses and damping factors from the quadratic pencil. To do this, we must invert equations (44), (49) and (50). Put $m_n = 1$, $m_i = u_i^2$, then

$$\left.\begin{aligned} g_i &= u_i a_i, & k_i &= u_i b_i, \\ G_i &= u_i^2 c_i - u_i a_i, & K_i &= u_i^2 \sigma_i^2 - u_i b_i, \end{aligned}\right\} i = 1, 2, \ldots, n-1, \tag{76}$$

$$g_n = c_n - \sum_{j=1}^{n-1} u_j a_j, \quad k_n = \sigma_n^2 - \sum_{j=1}^{n-1} u_j b_j. \tag{77,78}$$

The problem of finding positive u_i, $i = 1, 2, \ldots, n-1$, is a linear programming problem. The given spectral data will be realizable if the following problem has a solution:

$$u_i > 0, \quad u_i \geqslant a_i/c_i, \quad u_i > b_i/\sigma_i^2, \tag{79}$$

$$c_n - \sum_{j=1}^{n-1} a_j u_j \geqslant 0, \quad \sigma_n^2 - \sum_{j=1}^{n-1} b_j u_j > 0. \tag{80}$$

First, we consider the two lower bounds: a_j/c_j and b_j/σ_j^2. Suppose the jth mode is underdamped, then

$$\frac{a_j}{c_j} - \frac{b_j}{\sigma_j^2} = \frac{a_j\sigma_j^2 - b_j c_j}{c_j \sigma_j^2}, \tag{81}$$

and

$$a_j\sigma_j^2 - b_j c_j = \frac{R_j}{\sin\alpha_j}\{\sin\theta_j - 2\cos\alpha_j \sin(\alpha_j - \theta_j)\}. \tag{82}$$

The inequalities $0 \leqslant \theta_j < \pi$ imply $\sin\theta_j \geqslant 0$, and the inequalities $\pi/2 < \alpha_j < \pi$, $\alpha_j > \theta_j$ imply $-\cos\alpha_j \sin(\alpha_j - \theta_j) > 0$. Thus, $a_j\sigma_j^2 - b_j c_j > 0$. If the mode is overdamped, then

$$a_j\sigma_j^2 - b_j c_j = \frac{\mu_j' R_j + \mu_j^2 R_j'}{(\mu_j^{2'} - \mu_j^2)\mu_j \mu_j'} > 0. \tag{83}$$

Thus, in all cases

$$\frac{a_j}{c_j} > \frac{b_j}{\sigma_j^2}. \tag{84}$$

We must therefore take $u_j = (a_j/c_j) + x_j$, $x_j \geqslant 0, j = 1, \ldots, n-1$ so that inequalities (80) become

$$c_n - \sum_{j=1}^{n} \frac{a_j^2}{c_j} - \sum_{j=1}^{n-1} a_j x_j \geqslant 0, \tag{85}$$

$$\sigma_n^2 - \sum_{j=1}^{n} \frac{a_j b_j}{c_j} - \sum_{j=1}^{n-1} b_j x_j > 0. \tag{86}$$

The conditions for the existence of a non-negative solution $x_1, x_2, \ldots, x_{n-1}$ are thus

$$c_n - \sum_{j=1}^{n-1} \frac{a_j^2}{c_j} \geqslant 0, \quad \sigma_n^2 - \sum_{j=1}^{n-1} \frac{a_j b_j}{c_j} > 0. \tag{87}$$

The first condition states that **C** is ps–d. Inequality (84) shows that the second condition is somewhat stronger than the condition

$$\sigma_n^2 - \sum_{j=1}^{n-1} \frac{b_j^2}{\sigma_j^2} > 0 \tag{88}$$

which states that **K** is positive definite.

6. RECAPITULATION AND CONCLUSIONS

The procedure described in section 5 is somewhat involved, and it is difficult to see it as a whole. Therefore, we list the principal steps in the analysis, to see what we have established, and what remains to be established.

The data are the eigenvalues:

$$\lambda_j, \lambda'_j, j = 1, 2, \ldots, r, \quad \lambda_j, \bar{\lambda}_j, j = r+1, \ldots, n,$$

$$\mu_j, \mu'_j, j = 1, 2, \ldots, s, \quad \mu_j, \bar{\mu}_j, j = s+1, \ldots, n-1.$$

The data yield the quantities c_j, σ_j^2 through equation (51):

$$c_j = -\mu_j - \mu'_j, \quad \sigma_j^2 = \mu_j \mu'_j, \quad j = 1, 2, \ldots, s, \tag{89}$$

$$c_j = -\mu_j - \bar{\mu}_j, \quad \sigma_j^2 = \mu_j \bar{\mu}_j, \quad j = s+1, \ldots, n-1, \tag{90}$$

and similarly give the quantities d_j, ω_j^2 through equation (53):

$$d_j = -\lambda_j - \lambda'_j, \quad \omega_j^2 = \lambda_j \lambda'_j, \quad j = 1, 2, \ldots, r, \tag{91}$$

$$d_j = -\lambda_j - \bar{\lambda}_j, \quad \omega_j^2 = \lambda_j \bar{\lambda}_j, \quad j = r+1, \ldots, n. \tag{92}$$

Now $(c_j)_1^{n-1}$ and $(d_j)_1^n$ yield c_n through equation (56):

$$c_n = \sum_{j=1}^{n} d_j - \sum_{j=1}^{n-1} c_j. \tag{93}$$

Since $c_n > 0$, this equation yields the first necessary condition on the eigenvalue data:

$$-\sum_{j=1}^{r} (\lambda_j + \lambda'_j) - \sum_{j=r+1}^{n} (\lambda_j + \bar{\lambda}_j) + \sum_{j=1}^{s} (\mu_j + \mu'_j) + \sum_{j=s+1}^{n-1} (\mu_j + \bar{\mu}_j) > 0. \tag{94}$$

Now we pass to the next stage; the evaluation of a_j and b_j. From the eigenvalue data we form the quantities $R(\mu_j)$, $R(\mu'_j)$, $j = 1, 2, \ldots, s$. Then equations (69) give

$$a_j = \frac{R(\mu'_j) + R(\mu_j)}{\mu'_j - \mu_j}, \quad b_j = \frac{-\mu'_j R(\mu_j) - \mu_j R(\mu'_j)}{\mu'_j - \mu_j}, \quad j = 1, 2, \ldots, s. \tag{95}$$

These will be finite and positive if the overdamped eigenvalues satisfy inequalities (63) and (64), namely

$$\mu_1 < \mu_1' < \mu_2 < \mu_2' < \cdots < \mu_s' < 0, \tag{96}$$

$$\lambda_1 < \lambda_1' < \lambda_2 < \lambda_2' < \cdots < \lambda_r' < 0, \tag{97}$$

and, in addition, each μ_j, μ_j' lies between a λ_k, λ_k' pair, with one of each μ_j, μ_j' lying strictly between such a pair.

The corresponding formulae, for the a_j, b_j relating to underdamped modes, are given by equations (72)–(74):

$$\{R(\mu_i)\}^2 = R_j^2 \exp(2\mathrm{i}\theta_j), \quad 0 \leqslant \theta_j < \pi, \tag{98}$$

$$\mu_j = \rho_j \exp(\mathrm{i}\alpha_j), \quad \frac{\pi}{2} < \alpha_j < \pi, \tag{99}$$

$$a_j = \frac{R_j \sin\theta_j}{\rho_j \sin\alpha_j}, \qquad b_j = \frac{R_j \sin(\alpha_j - \theta_j)}{\sin\alpha_j}. \tag{100}$$

Now the simple interlacing conditions satisfied by the overdamped eigenvalues are replaced by inequalities which involve all the eigenvalues $(\lambda_j)_1^n$ and individual eigenvalues μ_j which appear in $R(\mu_j)$. The inequalities are those in (75), namely

$$0 \leqslant \theta_j < \alpha_j, \quad j = s+1, \ldots, n-1. \tag{101}$$

These conditions, and the corresponding interlacing conditions for the overdamped eigenvalues, ensure that the a_j, b_j are positive. Now equation (55) yields the last quantity, σ_n^2, appearing in the reduced matrix $\mathbf{Q}(\lambda)$:

$$\sigma_n^2 = \left\{ \prod_{j=1}^{n} \omega_j + \sum_{j=1}^{n-1} b_j^2 \prod_{j=1}^{n-1}{}' \sigma_k^2 \right\} \Bigg/ \prod_{j=1}^{n-1} \sigma_j^2. \tag{102}$$

At this point, we are assured that the data correspond to a real quadratic pencil, and we enter the last stage: the determination of the masses $m_i = u_i^2$. The analysis given in equations (76)–(87) shows that the inequalities (87) must hold, and if they do then we may take $x_j = \mathbf{0}$, i.e.,

$$m_n = 1, \quad m_j = u_j^2, \quad u_j = a_j/c_j. \tag{103}$$

Now for $j = 1, 2, \ldots, n-1$, we have

$$g_j = \frac{a_j^2}{c_j}, \quad k_j = \frac{a_j b_j}{c_j}, \quad G_j = 0, \quad K_j = \frac{a_j}{c_j}\left(\frac{a_j \sigma_j^2}{c_j} - b_j\right), \tag{104}$$

while

$$g_n = c_n - \sum_{j=1}^{n-1} a_j^2/c_j, \quad k_n = \sigma_n^2 - \sum_{j=1}^{n-1} a_j b_j/c_j. \tag{105}$$

If inequality (85) is strict, then we find an infinite family of other solutions with positive x_j corresponding to positive G_j, where now the parameters are given by more general equations (76)–(78) with $a_j = (a_j/c_j) + x_j$, $x_j > 0$, $j = 1, 2, \ldots, n-1$.

7. CONCLUSIONS

For a linear undamped vibrating system there is a strong and simple statement which can be made regarding the effect of a constraint: *the constrained eigenvalues interlace and unconstrained eigenvalues.* This paper has attempted to elucidate what happens when the system is subjected to (viscous) damping. Now the eigenvalues of both the original and constrained systems are complex. We have studied a particularly simple system with n degrees of freedom, with $(n-1)$ masses in parallel. We found that there are interlacing conditions on *overdamped* (negative real) eigenvalue pairs, but that the conditions on the complex pairs of unconstrained and constrained eigenvalues are not simple interlacing inequalities—they cannot be, because the eigenvalues lie in the complex plane, and the points in the plane cannot be ordered—but are more complicated inequalities involving all the eigenvalues at once.

Part 2 will discuss a theoretical–experimental study of the simplest case studied here, $n = 2$.

REFERENCES

1. F. P. GANTMAKHER and M. G. KREIN 1950 *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, Moscow Leningrad: State Publishing House for Technical-Theoretical Literature. English Translation by U.S. Atomic Energy Commission, Washington, DC, 1961.
2. G. H. GOLUB and D. BOLEY 1977 *Topics in Numerical Analysis*. In J. H. H. Miller editor, Inverse eigenvalue problems for band matrices. New York: Academic Press.
3. G. M. L. GLADWELL 1986 *Inverse Problems in Vibration*, Dordrecht: Martinus Nijhoff.
4. G. M. L. GLADWELL 1986 *Applied Mechanics Reviews* **39**, 1013–1018. Inverse problems in vibration I.
5. G. M. L. GLADWELL 1996 *Applied Mechanics Reviews* **48**, S25–S34. Inverse problems in vibration II.
6. G. M. L. GLADWELL and N. B. WILLMS 1988 *Inverse Problems* **4**, 1013–1024. Reconstruction of a tri-diagonal system from its frequency response at an interior point.
7. Y. M. RAM and J. CALDWELL 1992 *SIAM Journal of Applied Mathematics* **52**, 140–152. Physical parameters reconstruction of a free-free mass-spring system from its spectra.
8. Y. M. RAM 1993 *SIAM Journal of Applied Mathematics* **53**, 1762–1175. Inverse eigenvalue problems for a modified vibrating system.
9. G. M. L. GLADWELL 1997 *Inverse Problems* **13**, 311–322. Inverse vibration problems for finite element models **211**, 309–324.
10. G. M. L. GLADWELL 1999 *Journal of Sound and Vibration* **211**, 309–324. to appear, Inverse finite element vibration problems.
11. Y. M. RAM and S. ELHAY 1996 *SIAM Journal of Applied Mathematics* **56**, 232–244. Inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with application to damped oscillatory systems.
12. O. H. HALD 1976 *Linear Algebra and Applications* **14**, 63–85. Inverse eigenvalue problems for Jacobi matrices.